

Entropy of Isolated Horizons revisited

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Abstract

The decade-old formulation of the isolated horizon classically and within loop quantum gravity, and the extraction of the microcanonical entropy of such a horizon from this formulation, is reviewed, in view of recent renewed interest. There are two main approaches to this problem: one employs an $SU(2)$ Chern-Simons theory describing the isolated horizon degrees of freedom, while the other uses a reduced $U(1)$ Chern-Simons theory obtained from the $SU(2)$ theory, with appropriate constraints imposed on the spectrum of boundary states ‘living’ on the horizon. It is shown that both these ways lead to the same infinite series asymptotic in horizon area for the microcanonical entropy of an isolated horizon. The leading area term is followed by an unambiguous correction term logarithmic in area with a coefficient $-\frac{3}{2}$, with subleading corrections dropping off as inverse powers of the area.

1 Introduction

There appears to have been a resurgence in interest in the Loop Quantum Gravity approach towards black hole entropy. The main idea of this approach involves identifying a ‘boundary’ theory characterizing the degrees of freedom on an isolated horizon (of fixed cross-sectional area), consistent with the boundary conditions used to *define* such horizons [1] [2], and then counting the dimension of the Hilbert space of the quantum version of this boundary theory [3, 4, 5, 6]. This dimension is then considered to be the exponential of the *microcanonical* entropy of the isolated horizon. Clearly this is an effective field theory approach where the existence of an isolated horizon, as a null inner boundary of quantum space (on a spatial slice) punctured by spin network links in loop quantum gravity, is

assumed from the start, and not derived as a solution of the quantum Einstein equation (the Hamiltonian constraint, in a canonical description). Thus, one has to further make the assumption that the quantum Einstein equation does indeed yield spacetimes with this assumed property.

Before we begin, a word about our *notation* : spacetime forms which are also internal $SU(2)$ vectors are indicated with a boldface and an arrow on top. The exterior product between such forms includes the cross product between the internal vectors. The exterior product with an overdot indicates a standard exterior product between the spacetime forms and a scalar product between the internal $SU(2)$ vectors. For spacetime forms which are not internal vectors, indicated in boldface without arrow on top, the exterior product has the standard connotation. A plain cross-product operates between internal vector functions or between a function and a form which is also an internal vector.

Within a canonical formulation, vacuum general relativity is formulated on a partial Cauchy surface M in terms of the Barbero-Immirzi (BI) class of $SU(2)$ Lie-algebra valued connection one-forms expressed in the basis of $SU(2)$ generators $\vec{\mathbf{A}}$.¹ The canonically conjugate phase space variable is the $SU(2)$ -valued solder form, expressed in the basis of $SU(2)$ generators $\vec{\Sigma}$. In terms of these, the symplectic structure of vacuum spacetime takes the form (ignoring boundary terms)

$$\Omega_V(\delta_1, \delta_2) = \frac{1}{16\pi G} \int_M \delta_{[1} \vec{\mathbf{A}} \wedge \delta_2] \vec{\Sigma} . \quad (1)$$

We now introduce an isolated horizon as a null *inner* boundary of spacetime with fixed cross sectional area A_{IH} [1]. The boundary conditions do not lead to any obvious reduction in gauge invariance, so one expects the boundary theory to be a three dimensional topological gauge theory living on the null surface and having an $SU(2)$ gauge invariance. This theory is expected to be an $SU(2)$ Chern-Simons theory. An equivalent alternative description of the boundary theory in terms of a topological $SU(2)$ $\mathbf{B} - \mathbf{F}$ theory is also possible. We should perhaps mention that the use of $SU(2)$ Chern-Simons theory as a boundary theory to derive black hole entropy has a precedence: independently, Krasnov, Rovelli and Smolin [7] have considered this possibility, although not using the Isolated Horizon paradigm. The task of implementing the boundary conditions and deriving the phase space symplectic structure is nontrivial and has been accomplished with several simplifying assumptions, involving an additional gauge fixing [3], which reduces the gauge invariance to $U(1)$. As for any fixing of gauge, the convenience is always accompanied by additional constraints on the dynamical variables like the curvature; unfortunately, these constraints are not always implemented fully when deriving the symplectic structure. If they are taken into account, one expects to regain the full $SU(2)$ gauge invariance in phase space. Work is in progress to demonstrate this explicitly from generic isolated horizon

¹The reduction in gauge invariance from the full local Lorentz group ($SL(2, C)$) to the group of local rotations ($SU(2)$) is made by fixing the so-called time gauge whereby local Lorentz boosts are fixed on the spatial slice.

boundary conditions [8] (see, however, the recent preprint [9] which discusses an $SU(2)$ CS theory as the theory on an IH)).

In this short note, we first *assume* that the topological gauge theory on the isolated horizon is indeed an $SU(2)$ Chern-Simons theory with a coupling constant $k \equiv A_{IH}/8\pi\gamma l_P^2$. To be consistent with known properties of $SU(2)$ Chern-Simons theory, one assumes that $A_{IH} \gg l_P^2$ and the nearest integer value of k to the expression above is chosen. We then briefly review the derivation, given in 1998 by two of us, of the dimensionality of the Hilbert space of the quantum version of the theory [4]; in that paper $SU(2)$ *singlet states* are counted using the relation [10] between the Hilbert space of an $SU(2)$ Chern-Simons theory on a punctured $S^2 \times \mathbf{R}$ with the number of conformal blocks of the $SU(2)_k$ WZW model ‘living’ in that S^2 . We also review the further derivation by two of us [5] (given in 2000) showing how this leads to an infinite series asymptotic in A_{IH} for the microcanonical entropy, with a leading area term and corrections beginning with a term logarithmic in the area with a coefficient $-3/2$.

Next we do the counting of states differently : by first gauge fixing the $SU(2)$ to an $U(1)$ following the pioneering work of Ashtekar *et. al.* [3] using a covariantly constant internal vector \vec{r} . One then counts the states that are $U(1)$ invariant. However, we consider additionally the constraint on the $SU(2)$ curvature that the gauge fixing entails. At the level of the quantum theory, these additional constraints are shown to lead to precisely the same counting of states as recalled above in terms of $SU(2)$ invariant states. There is thus no discrepancy in the final answer for the microcanonical entropy.

2 Review of $SU(2)$ singlet state counting

In presence of the isolated horizon the bulk symplectic structure above is augmented by a boundary symplectic structure assumed to be given by that of an $SU(2)$ Chern-Simons theory,

$$\Omega_b(\delta_1, \delta_2) = \frac{k}{2\pi} \int_S \delta_{[1} \vec{A} \wedge \delta_2] \vec{A} \quad (2)$$

where $k \equiv A_{IH}/2\pi\gamma$ with γ being the Barbero-Immirzi parameter. Here S is the spatial S^2 foliation of the isolated horizon. On this S^2 the ‘Gauss law’ equation appropriate to local spatial rotations is

$$\frac{k}{2\pi} \vec{F} = - \vec{\Sigma} . \quad (3)$$

In eq. (3), both \vec{F} and $\vec{\Sigma}$ are pull backs of the curvature 2-form appropriate to the Barbero-Immirzi connection and the solder form on partial Cauchy surface M to S .

In LQG, the Hilbert space is assumed to be the tensor product $\mathcal{H}_V \otimes \mathcal{H}_{IH}$ corresponding to the bulk spin network space and the isolated horizon respectively. The geometric

variables in (3) above become operators acting on appropriate Hilbert spaces. The solder form operator has an action on spin network bulk states as eigenstates with support only on the punctures of S where spin network links pierce it. Consequently, acting on the isolated network boundary states, the curvature operator has the action

$$\mathcal{I} \otimes \frac{k}{2\pi} \hat{\vec{F}}(x) |\psi_V\rangle \otimes |\chi_{IH}\rangle = - \sum_p \delta^{(2)}(x, x_p) {}^2\epsilon_p \vec{T}_p |\psi_V\rangle \otimes |\chi_{IH}\rangle \quad (4)$$

where, the sum is over a set of punctures carrying $SU(2)$ spin representation of the generators \vec{T}_p (corresponding to spin j_p) at the p th puncture, and ${}^2\epsilon_p$ is the area 2-form for that puncture. Given that upto $O(l_P^2)$, the sum of these areas over the entire set of punctures must equal the fixed classical area A_{IH} , one immediately realizes that the set of states to be counted must obey the constraint that they are $SU(2)$ *singlets*.

This counting has been accomplished in [4]. One utilizes the connection [10] between the dimensionality of the Hilbert space of $SU(2)$ Chern-Simons theory living on a punctured S^2 ($\times \mathcal{R}$) and the number of *conformal blocks* of the boundary two dimensional conformal field theory – $SU(2)_k$ WZW model on the punctured S^2 . One also makes use of the fusion algebra and the Verlinde formula for the representation matrices of that algebra. In terms of the spins j_1, j_2, \dots, j_p on punctures, the dimension of the space of $SU(2)$ -singlet boundary states is (for $k \rightarrow \infty$)

$$\mathcal{N}(j_1, \dots, j_p) = \prod_{n=1}^p \sum_{m_n=-j_n}^{j_n} \left(\delta_{m_1+\dots+m_p,0} - \frac{1}{2} \delta_{m_1+\dots+m_p,-1} - \frac{1}{2} \delta_{m_1+\dots+m_p,1} \right). \quad (5)$$

The last two terms precisely ensure that the counting is restricted to $SU(2)$ singlet boundary states, since these alone obey the ‘Gauss law constraint’ which ensures local gauge invariance or ‘physicality’ of the counted states.

To extract the microcanonical entropy of the isolated horizon, one may follow our work [5]; the entropy turns out to be

$$S_{IH} = S_{BH} - \frac{3}{2} \log S_{BH} + \text{const.} + O(S_{BH}^{-1}), \quad (6)$$

where, S_{BH} is the usual Bekenstein-Hawking area expression for the entropy: $S_{BH} = A_{IH}/4l_P^2$. In this, the Barbero-Immirzi parameter has been ‘fitted’ to agree with the correct normalization of the Bekenstein-Hawking area term. There is absolutely no other ambiguity in this infinite series, each of whose terms are finite and calculable.

3 The $U(1)$ counting

The implementation of the isolated horizon boundary conditions and derivation of the boundary symplectic structure has been accomplished at the classical level in [1] and

later follow-up work [2]. It has been claimed that this is most facile if one makes a further fixing of the gauge invariance on a Cauchy surface from $SU(2)$ to a residual $U(1)$ invariance generated by the diagonal $SU(2)$ generator alone. This is done by picking an internal $SU(2)$ vector field \vec{r} which is covariantly constant on the S^2 foliation of an isolated horizon (or equivalently an $su(2)$ -valued function which is covariantly constant on the S^2). It is obvious such an internal vector field always exists on the S^2 .

$$D\vec{r} \equiv d\vec{r} + \vec{\mathbf{A}} \times \vec{r} = 0. \quad (7)$$

where $\vec{\mathbf{A}}$ is pull-back to the S^2 of the $SU(2)$ BI connection

$$\vec{\mathbf{A}} = \vec{r}\mathbf{B} + \vec{\mathbf{C}} \quad (8)$$

with,

$$\begin{aligned} \vec{r} \cdot \vec{\mathbf{C}} &= 0, \quad \vec{r}^2 = 1 \\ D\vec{r} &= d\vec{r} + \vec{\mathbf{C}} \times \vec{r} = 0. \end{aligned} \quad (9)$$

Observe that one can solve the second equation above explicitly for $\vec{\mathbf{C}}$

$$\vec{\mathbf{C}} = -\vec{r} \times d\vec{r}. \quad (10)$$

The pullback of the curvature two-form to the S^2 is

$$\begin{aligned} \vec{\mathbf{F}} &= d\vec{\mathbf{A}} + \frac{1}{2}\vec{\mathbf{A}} \wedge \vec{\mathbf{A}} \\ &= \vec{r} \left(d\mathbf{B} - \frac{1}{2}\vec{r} \cdot d\vec{r} \wedge d\vec{r} \right) \end{aligned} \quad (11)$$

The second line of eq. (11) follows from the first using the decomposition (8) and the solution (10), together with the observation that the 2-form $d\vec{r} \wedge d\vec{r} = \vec{r} \cdot \vec{r} \cdot d\vec{r} \wedge d\vec{r}$ which ensues because $\vec{r} \times d\vec{r} \wedge d\vec{r} = 0$.

The projection of this curvature along \vec{r} is given by

$$\mathbf{f} \equiv \vec{r} \cdot \vec{\mathbf{F}} = d\mathbf{B} - \frac{1}{2}\vec{r} \cdot d\vec{r} \wedge d\vec{r}. \quad (12)$$

The second term in eq. (12) is actually a winding number density associated with maps from S^2 to S^2 ; if we write it as $-d\Omega$, then

$$\frac{1}{8\pi} \int_S d\Omega = N \in \mathcal{Z}. \quad (13)$$

Thus, we may write the $U(1)$ curvature as

$$\mathbf{f} = d\mathbf{B}' \quad (14)$$

where, $\mathbf{B}' \equiv \mathbf{B} - \mathbf{\Omega}$. We note that for the quantum isolated horizon, the $U(1)$ connection \mathbf{B}' vanishes locally on the S^2 , except on the punctures. Because of the nontrivial winding at each puncture, it is a nontrivial $U(1)$ bundle on S^2 . This is the contribution that accumulates to giving the counting of states leading to the microcanonical entropy in this approach.

The counting now proceeds by solving the $U(1)$ projected version of eq. (3)

$$\frac{k}{2\pi} \mathbf{f} = -\vec{r} \cdot \vec{\Sigma} \quad (15)$$

which immediately translates, in the quantum version of the theory to

$$\mathcal{I} \otimes \frac{k}{2\pi} \hat{\mathbf{f}}(x) |\psi_V\rangle \otimes |\chi_{IH}\rangle_{U(1)} = - \sum_p \delta^{(2)}(x, x_p)^2 \epsilon_p \vec{r} \cdot \vec{T}_p |\psi_V\rangle \otimes |\chi_{IH}\rangle_{U(1)} , \quad (16)$$

which implies that the states to be counted are $U(1)$ Chern-Simons theory states on the punctured sphere with the net spin projection along \vec{r} vanishing: $\vec{r} \cdot \sum_p \vec{T}_p = 0$. Observe that one can always rotate \vec{r} locally so that this is possible, even though globally this vector corresponds to a nontrivial $U(1)$ bundle on S .

This $U(1)$ counting has been done in a variety of ways [11]. If, in the reduced $U(1)$ theory, we disregard the consequences of the constraint implied for the projection of $SU(2)$ field strength orthogonal to the direction of vector \vec{r} (see the next paragraph), the final result for the dimensionality of the $U(1)$ Chern-Simons Hilbert space is given by the first term of the eqn. (5). For macroscopic ($A_{IH} \gg l_P^2$) isolated horizons the corresponding microcanonical entropy is given by

$$S_{IH} = S_{BH} - \frac{1}{2} \log S_{BH} + \dots . \quad (17)$$

The leading term once again offers a fit to the BI parameter, which is the same as in the $SU(2)$ case, if spins at all punctures are chosen to be $1/2$. The most obvious difference though is the appearance of a logarithmic LQG correction to the Bekenstein-Hawking area term, *with a coefficient $-1/2$ instead of $-3/2$* as found above by doing the $SU(2)$ singlet counting. This is obviously because of the additional gauge fixing performed in implementing the isolated horizon boundary conditions; the diagonal $SU(2)$ generator is taken parallel to the covariantly constant internal vector field \vec{r} chosen above. Thus, the generators orthogonal to \vec{r} are set to zero, and hence the apparent discrepancy between (17) and (6).

However, observe that the curvature given in (11) has *vanishing* projection orthogonal to \vec{r} , i.e.,

$$\vec{\mathbf{F}} \times \vec{r} = 0 . \quad (18)$$

This can also be derived independently because of the gauge choice in terms of the special internal vector \vec{r} obeying $D(\vec{\mathbf{A}})\vec{r} = 0$; one obtains the same constraint

$$[D_a, D_b]\vec{r} = 0 = \vec{F}_{ab} \times \vec{r} \quad (19)$$

where a, b are spacetime indices on S^2 . This constraint arises as an essential and *inevitable* part of the additional gauge fixing performed on the theory on S , reducing the residual invariance on S (in time gauge) from $SU(2)$ to $U(1)$. The constraint imposes a direct and very significant additional restriction on the class of ‘physical’ states contributing to the microcanonical entropy, over and above that of $U(1)$ -neutrality. If we use eq. (3) and consider the quantum version of the above additional constraint on the spin network bulk and boundary states, we obtain,

$$\sum_p \delta^{(2)}(x, x_p) {}^2\epsilon_p \vec{r} \times \vec{T}_p |\psi_V\rangle \otimes |\chi_{IH}\rangle_{U(1)} = 0 \quad (20)$$

where \vec{T} are the $su(2)$ generators. One must now count the dimension of IH states that satisfy the additional constraint (20) apart from $U(1)$ neutrality. *This, unfortunately, has not been done in the later literature on $U(1)$ counting approaches [11].*

Consider now the behaviour of eq. (16) under the action of the $SU(2)$ Gauss law constraint. Under the action of this constraint the states are annihilated. It is easy to check that the LHS of (16) is actually invariant under $SU(2)$ gauge transformations (for a fixed \vec{r} , as in this case) generated by the Gauss law constraint, provided one makes use of the operator version of (18). More specifically, under an $SU(2)$ gauge transformation, $\vec{\mathbf{F}} \rightarrow \vec{\mathbf{F}}' = \vec{\mathbf{F}} + \vec{\theta} \times \vec{\mathbf{F}}$ where $\vec{\theta}$ is an infinitesimal transformation parameter. Note though that the $U(1)$ field strength \mathbf{f} does not change, because of (18)

$$\begin{aligned} \mathbf{f}' &= \vec{r} \cdot \vec{\mathbf{F}} + \vec{r} \cdot \vec{\theta} \times \vec{\mathbf{F}} \\ &= \mathbf{f} + \vec{\theta} \cdot \vec{\mathbf{F}} \times \vec{r} = \mathbf{f} \end{aligned} \quad (21)$$

The LHS of (16) thus remains invariant under local $SU(2)$ gauge transformations. There is a small subtlety in the manner in which the RHS of (16) also remains invariant under such. Observe that such transformations must not remove the system off the gauge surface defined by the gauge fixing condition (9), i.e., $D\vec{r} = 0$, valid on the 2-sphere foliation of the IH. This imposes the restriction on the $SU(2)$ gauge parameter $\vec{\theta}$ to lie along the fixed internal vector \vec{r} , i.e., $\vec{\theta} = \vec{r}\alpha$ on such 2-sphere foliations. The parameter can of course be arbitrary away from the 2-spheres. It is easy to see that the RHS of (16) is indeed invariant under such restricted $SU(2)$ transformations, as also is the additional constraint eq. (20) on the IH states, both being relations valid on the 2-sphere foliations S .

To conclude, note that the import of eqn. (20) is to imply that spins on punctures should be such that the net spin orthogonal to the direction \vec{r} is zero. This then, along with net $U(1)$ neutrality, is the requirement that all admissible isolated horizon states

contributing to the microcanonical entropy are $SU(2)$ singlets. This leads to the LQG result for the entropy as written out in the formula (5), where the first term on the right-hand side counts the $U(1)$ neutral states and the last two terms subtract out the overcounted states with the net azimuthal quantum number equal to zero coming from net non-zero spin (1, 2, 3,) states which need to be excluded to impliment the additional constraint (20). Thus recalling (6), even in this counting the leading logarithmic LQG correction has a coefficient $-\frac{3}{2}$ as advertized in the abstract and in the introduction.

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